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ON FORCED OSCILLATIONS OF LINEAR ELASTIC SYSTEMS WITH DAMPING*

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The concept of a transmission coefficient, which has been introduced for an elastic system with one degree of freedom /1-2/, is generalized to the case of an arbitrary linear elastic system of finite mass to which a linear visco-elastic absorber of general form is attached. For an elastic system with one degree of freedom the problem of the smallest absorber mass that gives a specified transmission coefficient is solved.

Various optimality criteria for the parameters of an oscillation absorber have been considered, the most common one being the minimization of the transmission coefficient of the forcing term which takes a value that is a maximum in some frequency domain (perhaps the entire spectrum). Elastic systems with several degrees of freedom or absorbers have been investigated /3-5/, but only special cases have as a rule been solved.

We shall consider an elastic system of finite mass M, occupying a known volume V and possessing known elastic and inertial properties. We will first specify the influence matrix $K(N, N')$ and secondly the generalized density matrix $\rho(N)$ (N and N' are points in the volume V, with $K(N, N') = K^T(N', N)$ and $\rho(N) = \rho^T(N)$). In general the matrices K and ρ are of third order; their form depends on the system of coordinates. For a given elastic system there exists a spectrum of eigenfrequencies $\{\Omega_j\}$ and eigenfunctions $\{\mathbf{u}_j(N)\}$ ($j = 1, 2, \ldots, S$; $S \leqslant \infty$ is the number of degrees of freedom of the elastic system), satisfying the conditions

$$\Omega_j^2 \int_V K(N, N') \rho(N') \mathbf{u}_j(N')\, dV(N') = \mathbf{u}_j(N)$$

$$\int_V \mathbf{u}_j^T(N) \rho(N) \mathbf{u}_k(N)\, dV(N) = M\delta_{jk}$$

$$1 \leqslant j,\ k \leqslant S$$

(where δ_{jk} is the Kronecker delta and $dV(N)$ is the volume element at the point N; from now on vectors are assumed to be columns). The influence matrix can be put in the form /6/

$$K(N, N') = M^{-1} \sum_{k=1}^{S} \frac{\mathbf{u}_k(N)\mathbf{u}_k^T(N')}{\Omega_k^2}$$

It describes the displacement of points of the elastic system under the action of a constant force applied to it. If this force is of the form $e^{\lambda t}\mathbf{F}(N')$ per unit volume, where λ is the characteristic index, (the most important practical case being when the quantity λ is purely imaginary), then a particular solution of the integrodifferential equation of motion of the elastic system, derived in /7/, has the form $e^{\lambda t}\mathbf{U}(N)$, where

$$\mathbf{U}(N) = \int_V K_1(N, N', \lambda)\mathbf{F}(N')\, dV(N') \tag{1}$$

is the displacement (both here and below to be taken relative to the equilibrium position) of the point N, while

$$K_1(N, N', \lambda) = M^{-1} \sum_{k=1}^{S} \frac{\mathbf{u}_k(N)\mathbf{u}_k^T(N')}{\Omega_k^2 + \lambda^2}$$

is the dynamic influence matrix, ($K_1 = K$ when $\lambda = 0$).

We assume that a linear visco-elastic absorber is attached to the elastic system at the points $L_1, L_2, \ldots, L_R$; for $j = 1, 2, \ldots, R$ the point L_j can be displaced in the direction of the unit vector $\mathbf{n}_j$ (Fig.1). Suppose furthermore that a lumped force $e^{\lambda t}\mathbf{F}_0$ is applied to

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a given point N', where $\mathbf{F}_0$ is constant. We shall find the displacement of an arbitrary point N under the action of this force and the reaction of the absorber. The required displacement should be of the form $e^{\lambda t}\mathbf{U}(N)$; we have to find the function $\mathbf{U}(N)$.

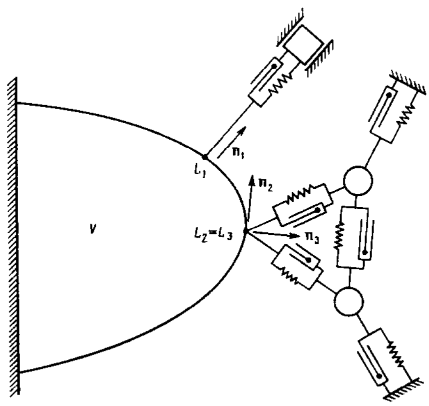

Fig.1

The total force acting on the elastic system is equal to

$$\mathbf{F}(N,t) = e^{\lambda t}\left[\mathbf{F}_0\delta(N,N') + \sum_{j=1}^{R} F_{1j}\mathbf{n}_j\delta(N,L_j)\right] \tag{2}$$

where δ is the Dirac delta function and F_{1j} is a force to be defined, acting at the point L_j in the direction $\mathbf{n}_j$ $(1 \leqslant j \leqslant R)$. Substituting expression (2) into (1), we find that

$$\mathbf{U}(N) = K_1(N,N',\lambda)\mathbf{F}_0 + \sum_{j=1}^{R} K_1(N,L_j,\lambda)\mathbf{n}_j F_{1j} \tag{3}$$

If $x_{0j} = \mathbf{n}_j{}^T\mathbf{U}(L_j)$ is the projection of the displacement of the point L_j onto $\mathbf{n}_j$ $(1 \leqslant j \leqslant R)$, then it follows from equality (3) that

$$\mathbf{n}_j{}^T K_1(L_j,N',\lambda)\mathbf{F}_0 + \sum_{k=1}^{R} \mathbf{n}_j{}^T K_1(L_j,L_k,\lambda)\mathbf{n}_k F_{1k} = x_{0j} \tag{4}$$

If $v_{jk} = \mathbf{n}_j{}^T\mathbf{u}_k(L_j)$ for $1 \leqslant j \leqslant R$, $1 \leqslant k \leqslant S$, and $\mathbf{v}_k$ is an R-dimensional vector with components $v_{1k}, v_{2k}, \ldots, v_{Rk}$, then (4) can be written as

$$Q(N',\lambda)\mathbf{F}_0 + G(\lambda)\mathbf{F}_1 = \mathbf{x}_0 \tag{5}$$

$$\mathbf{x}_0 = \{x_{01}, x_{02}, \ldots, x_{0R}\},\quad \mathbf{F}_1 = \{F_{11}, F_{12}, \ldots, F_{1R}\}$$

$$Q(N,\lambda) = M^{-1}\sum_{k=1}^{S}\frac{\mathbf{v}_k\mathbf{u}_k{}^T(N)}{\Omega_k{}^2+\lambda^2}$$

$$G(\lambda) = M^{-1}\sum_{k=1}^{S}\frac{\mathbf{v}_k\mathbf{v}_k{}^T}{\Omega_k{}^2+\lambda^2}$$

(the matrix Q has R rows and a number of columns equal to the dimension of the vector $\mathbf{F}_0$, in the general case equal to 3; the matrix G is square and of order R).

However, $F_1 = -\Phi(\lambda)\mathbf{x}_0$, where $\Phi(\lambda)$ is the transmission matrix of the absorber (of order R; it defines the forces acting on the points of attachment of the absorber due to their displacement; this displacement and the forces caused by them should be proportional to $e^{\lambda t}$ /7/. It then follows from relation (5) that

$$x_0 = [E + G(\lambda)\Phi(\lambda)]^{-1}Q(N',\lambda)F_0 \tag{6}$$

$$F_1 = -\Psi(\lambda)Q(N',\lambda)F_0,\quad \Psi(\lambda) = \Phi(\lambda)[E + G(\lambda)\Phi(\lambda)]^{-1}$$

(E is the unit matrix of order R), while the displacement of an arbitrary point N under the action of the force

$$e^{\lambda t}\sum_{j=1}^{R} F_{1j}\mathbf{n}_j\delta(N, L_j)$$

$(F_{11}, F_{12}, \ldots, F_{1R}$ are defined in (6)) is equal to $e^{\lambda t}\mathbf{U}_0(N)$ where

$$\mathbf{U}_0(N) = \sum_{j=1}^{R} K_1(N, L_j, \lambda)\,\mathbf{n}_j F_{1j} = M^{-1}\sum_{j=1}^{R} F_{1j}\sum_{k=1}^{S}\frac{\mathbf{u}_k(N)\,v_{jk}}{\Omega_k^2+\lambda^2} =$$
$$Q^T(N,\lambda)\,\mathbf{F}_1 = -Q^T(N,\lambda)\,\Psi(\lambda)\,Q(N',\lambda)\,\mathbf{F}_0$$

The displacement of the point N under the combined action of the $e^{\lambda t}\mathbf{F}(N')$ force per unit volume and the absorber reaction is equal to $e^{\lambda t}\mathbf{U}(N)$, where

$$\mathbf{U}(N) = \int_V H(N, N', \lambda)\,\mathbf{F}(N')\,dV(N) \qquad (7)$$
$$H(N, N', \lambda) = K_1(N, N', \lambda) - Q^T(N, \lambda)\,\Psi(\lambda)\,Q(N', \lambda)$$

The matrix H is a generalization of the influence matrix.

The following can serve as an example of an optimization problem: an absorber of given construction and least possible mass is to be attached to an elastic system with given elastic and inertial properties so that the quantity

$$\max \| H(N, N', i\Omega) \|$$

has a given magnitude; this maximum is taken over all points N and N' of the elastic system and frequencies Ω within some given domain, (and in special cases, over the entire spectrum). As a norm one can take, for example, the largest of the moduli of the elements of the matrix H. In its general form this problem is very difficult.

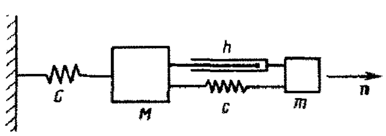

Fig.2

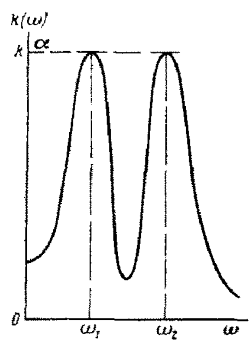

Fig.3

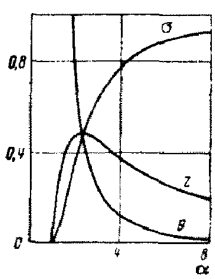

Fig.4

We will consider as an example a point mass M, attached to a fixed base by a spring of stiffness C; this mass is connected to an absorber in the form of a point mass m by means of a spring of stiffness c and a dashpot with viscous friction coefficient h; both masses (M and m) can be displaced in the direction of the unit vector $\mathbf{n}$ (Fig.2). In this system $R = S = 1$ and all matrices (K_1, Q, G, Φ, H) turn into functions of the characteristic index λ. This only eigenfrequency of this elastic system without the absorber is $\Omega_1 = (C/M)^{1/2}$, $v_{11} = 1$, $K_1(\lambda) = Q(\lambda) = G(\lambda) = (C + M\lambda^2)^{-1}$, $\Phi(\lambda) = m\lambda^2(h\lambda + c)(m\lambda^2 + h\lambda + c)^{-1}$ (see /7/), while according to formula (7)

$$H(\lambda) = (M\Omega_1^2)^{-1}\{(r^2 + zr + \sigma)\,[(r^2+1)(r^2 + zr + \sigma) + \theta r^2(zr + \sigma)]^{-1}\} \qquad (8)$$
$$(\theta = m/M,\ z = h/(m\Omega_1),\ \sigma = c/(m\Omega_1^2),\ r = \lambda/\Omega_1)$$

The expression in curly brackets on the right-hand side of (8) is equal to the transmission coefficient introduced in /1, 2/; the dimensionless matrix-function

$$M\Omega_1^2 H(N, N', \lambda),$$

can serve as a generalization of this coefficient to the case of an arbitrary elastic system, H being defined by formula (7).

Applied to the problem under consideration, the optimization problem formulated above can be stated as follows: it is required to find the least possible value of θ together with Z and σ such that for all real ω the quantity

$$k(\omega)=|(\omega^2+iz\omega-\sigma)[(\omega^2-1)(\omega^2+iz\omega-\sigma)+\theta\omega^2(iz\omega-\sigma)]^{-1}|$$

is never greater than some specified α. From the approximate solution derived in /2/ it follows that an analytical criterion for optimality defined as above in terms of θ, z and σ, is the presence in the function $k(\omega)$ of two smooth maxima equal to α at the point $\omega=\omega_1$ and $\omega=\omega_2$ (Fig.3). The conditions

$$k(\omega_-)=k(\omega_+)=\alpha,\ dk/d\omega(\omega_-)=dk/d\omega(\omega_+)=0 \tag{9}$$

are a system of four equations with five unknowns: $\omega_-, \omega_+, \theta, z$ and σ; the missing equation is the minimum condition on θ. The exact solution of this system is extremely difficult.

One can express ω_- and ω_+ in terms of new unknowns ω_0 and ε:

$$\omega_\pm=\omega_0(1\pm\varepsilon)^{1/2} \tag{10}$$

and show that for $\alpha\gg 1$ a minimum of θ is reached if $\varepsilon\approx\alpha^{-1}$. Numerical analysis shows that as α increases the quantity θ, found by solving system (9), depends more weakly on ε. Hence one can assume that for all α the quantities $\omega_\pm$ can be defined by formulae (10) with $\varepsilon=\alpha^{-1}$.

Graphs of the values of θ, z and σ computed on the basis of this condition as functions of α are shown in Fig.4. For large α we have $\theta\approx 2\alpha^{-2}, z\approx\sqrt{3}\alpha^{-1}$, and $\sigma\approx 1-4\alpha^{-2}$; for $\alpha\approx 1$ we have $\theta\approx 2(\sqrt{5}-1)(\alpha^{-2}-1)^{-1}$, $z\approx\sqrt{2}(\sqrt{5}+1)(\alpha^2-1)/8$, and $\sigma\approx(\alpha^2-1)^2/8$. The excess of θ over the value $2(\alpha^2-1)^{-1}$ obtained from the approximate solution /2/ tends to 24% as $\alpha\to 1+0$, but already at $\alpha=1.1$ it is 9%, while for $\alpha=2.5$ it does not exceed 1%.

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